Example 1. Simplify: (a) $x^{\frac{2}{3}} \cdot x^{\frac{5}{6}}$, (b) $(x^{-4})^{-\frac{3}{4}}$.

Solution:

(a) Applying law (1) we have:

$$x^{\frac{2}{3}} \cdot x^{\frac{5}{6}} = x^{\frac{2}{3}+\frac{5}{6}} = x^{\frac{9}{6}} = x^{\frac{3}{2}}.$$

(b) Applying law (2), we have:

$$(x^{-4})^{-\frac{3}{4}} = x^{(-4)(-\frac{3}{4})} = x^3.$$

Note that in the above operations no attempt is made to interpret the fractional exponents as equivalent radicals. Our rules for working with exponents allow us to operate directly on the given quantities, and the introduction of the radical equivalents would merely make the operations more awkward to carry out.

Example 2. Simplify: $\sqrt[3]{8\,x^3y^6} \div \sqrt{xy^{-2}}$.

Solution: $\sqrt[3]{8\,x^3y^6} \div \sqrt{xy^{-2}} = 2\,xy^2 \div x^{\frac{1}{2}}y^{-1}$

$= 2\,xy^2 \cdot x^{-\frac{1}{2}}y^1$ [Division by $x^{\frac{1}{2}}y^{-1}$ is equivalent to multiplication by $x^{-\frac{1}{2}}y^1$.]

$= 2\,x^{\frac{1}{2}}y^3$ [Adding exponents for the powers of x and for the powers of y.]

Example 3. Simplify: $2\,x^{-\frac{1}{2}} \cdot (x^{-\frac{1}{2}} + x^{\frac{1}{2}})^{-1}$.

Solution: Note particularly the binomial $x^{-\frac{1}{2}} + x^{\frac{1}{2}}$ raised to power -1. The only valid interpretation of this quantity is

$$\frac{1}{x^{-\frac{1}{2}} + x^{\frac{1}{2}}}.$$

Thus, we have:

$$2\,x^{-\frac{1}{2}} \cdot (x^{-\frac{1}{2}} + x^{\frac{1}{2}})^{-1} = \frac{2\,x^{-\frac{1}{2}}}{x^{-\frac{1}{2}} + x^{\frac{1}{2}}} = \frac{\frac{2}{x^{\frac{1}{2}}}}{\frac{1}{x^{\frac{1}{2}}} + x^{\frac{1}{2}}} = \frac{2}{1+x}.$$

(Multiplying each of the three terms in the fraction by $x^{\frac{1}{2}}$)

Exercises [A-1]

In exercises 1–8, select the correct definition from (a), (b), and (c).

	(a)	(b)	(c)
1. $x^{\frac{1}{2}}$ is defined as	$\frac{1}{2}\,x$	$\frac{2}{x}$	$\sqrt{x}$
2. x^0 is defined as	0	1	$\frac{x}{0}$
3. x^{-1} is defined as	$-x$	$\frac{-1}{x}$	$\frac{1}{x}$
4. $x^{\frac{2}{3}}$ is defined as	$\sqrt[3]{x^2}$	$\sqrt{x^3}$	$\frac{2}{3\,x}$

	(a)	(b)	(c)
5. x^{-2} is defined as	$-2x$	$\frac{1}{x^2}$	$\frac{1}{\sqrt{x}}$
6. $x^{-\frac{1}{2}}$ is defined as	$\frac{1}{\sqrt{x}}$	$\frac{1}{\frac{1}{2}x}$	$-2x$
7. $2x^{-1}$ is defined as	$-2x$	$\frac{1}{2x}$	$\frac{2}{x}$
8. $\left(\frac{3}{x}\right)^{-2}$ is defined as	$-6x$	$\frac{x^2}{9}$	$\frac{2x}{3}$

In exercises 9–15, write the numerical values without exponents or radicals.

	(a)	(b)	(c)	(d)	(e)
9.	3^2	$9^{\frac{1}{2}}$	$(\frac{1}{3})^{-2}$	$9^{-\frac{1}{2}}$	9^0
10.	2^3	$8^{\frac{1}{3}}$	$(\frac{1}{2})^{-3}$	$8^{-\frac{1}{3}}$	$(-8)^{\frac{1}{3}}$
11.	2^{-4}	$(\frac{1}{2})^4$	$4^{\frac{1}{2}}$	$(\frac{1}{4})^{-\frac{1}{2}}$	$(\frac{1}{2})^{-1}$
12.	$8^{\frac{2}{3}}$	$16^{\frac{3}{4}}$	$27^{-\frac{1}{3}}$	27^0	$2(7^0)$
13.	$4^{-\frac{3}{2}}$	$64^{\frac{2}{3}}$	$(\frac{1}{27})^{-\frac{2}{3}}$	$16^{\frac{1}{2}}$	$25^{\frac{3}{2}}$
14.	$(\frac{1}{3})^{-3}$	$(\frac{9}{16})^{-\frac{1}{2}}$	$(2\frac{1}{4})^{\frac{1}{2}}$	$(\frac{8}{27})^{\frac{2}{3}}$	$(\frac{1}{4})^{-2\frac{1}{2}}$
15.	$(-2)^{-2}$	$(-\frac{1}{2})^{-3}$	$100^{1\frac{1}{2}}$	$(-3\frac{3}{8})^{-\frac{1}{3}}$	$(1\frac{9}{16})^{-\frac{1}{2}}$

Simplify:

	(a)	(b)	(c)	(d)
16.	$x^6 \cdot x^{-2}$	$x^6 \div x^2$	$(x^6)^{\frac{1}{2}}$	$\sqrt{x^6}$
17.	$x^8 \cdot x^{-5}$	$x^8 \div x^5$	$(x^{27})^{\frac{1}{3}}$	$\sqrt[3]{x^{27}}$
18.	$x^{-3} \cdot x^{-3}$	$(x^{-3})^2$	$(x^{-3})^{\frac{1}{3}}$	$\sqrt[3]{\frac{1}{x^3}}$
19.	$\left(\frac{1}{x}\right)^{-1}$	$(x^{-1})^{-1}$	$(x^{-3})^{-\frac{1}{3}}$	$(27x^{-3})^{-\frac{1}{3}}$
20.	$x^{\frac{1}{2}} \cdot x^{\frac{1}{2}}$	$x^{\frac{2}{3}} \cdot x^{\frac{2}{3}} \cdot x^{\frac{2}{3}}$	$(x^{\frac{3}{4}})^4$	$(2x^{-\frac{1}{2}})^{-2}$

21. (a) $x^{\frac{3}{2}} \cdot x^{\frac{1}{2}} \cdot x^{-3}$; (b) $2x^{-\frac{1}{2}} \cdot 3x^{-2} \cdot x^3$; (c) $x^{\frac{2}{3}} \cdot x^{\frac{3}{2}} \div 2x^{\frac{1}{6}}$

22. (a) $(3x^{\frac{1}{2}})^2 \cdot (2x^{\frac{1}{3}})^{-3}$; (b) $(8x^{-3})^{\frac{1}{3}} \div 3x^{-1}$

23. (a) $x^{\frac{1}{2}} \cdot x^{-\frac{1}{2}}$; (b) $x^{\frac{1}{2}}(x^{\frac{1}{2}} + x^{-\frac{1}{2}})$; (c) $x^{\frac{3}{2}}(x^{\frac{1}{2}} - 2x^{-\frac{1}{2}} + x^{-\frac{3}{2}})$

24. (a) $x^{\frac{1}{2}} \div x^{-\frac{1}{2}}$; (b) $\dfrac{x^{\frac{1}{2}} + x^{-\frac{1}{2}}}{x^{-\frac{1}{2}}}$; (c) $\dfrac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{2x^{\frac{1}{2}}}$

25. (a) $x^{-1} + 2x^{-2}$ (b) $\dfrac{2x^{-2} - x^{-1}}{2x^{-2}}$ (c) $\dfrac{2^{-1}}{2^{-1} + 2^{-2}}$

26. Find the value of x for which $x^{\frac{3}{2}} = 8$.
27. Find the value of x for which $5\,x^{\frac{1}{2}} = 20$.
28. Solve: (a) $2\,x^{-2} = 18$; (b) $6\,x^{-\frac{1}{2}} = 2$.
29. Evaluate the expression $\left(\dfrac{x^{\frac{1}{2}} + 2\,x^{-\frac{1}{2}}}{x^{-\frac{1}{2}}}\right)^{\frac{1}{3}}$ when $x = 25$.
30. Simplify: $(2\,a^{-1}b)^{-2} \cdot (\frac{1}{4}\,a^3b^{-3})^{-\frac{1}{2}}$.

Exercises [A–2]

1. When $x = \frac{1}{4}$, evaluate (a) $2\,x^{-1}$; (b) $(2\,x)^{-1}$; (c) $(2\,x^{\frac{1}{2}})^{-4}$.
2. Evaluate: (a) 2^{-3}; (b) $100^{\frac{1}{2}}$; (c) $16^{-\frac{3}{4}}$; (d) $(\frac{1}{4})^{-\frac{1}{2}}$.
3. Evaluate: (a) $3^0 \div 3^{-2}$; (b) $2^{-3} \div 2$; (c) $10^{\frac{3}{4}} \div 10^{\frac{3}{4}}$.
4. Simplify: (a) $(16\,p^4q^2)^{\frac{1}{2}}$; (b) $(27\,a^3b^{-6})^{\frac{1}{3}}$; (c) $\left(\dfrac{a^4}{4\,b^2}\right)^{-\frac{3}{2}}$.
5. Evaluate $(x^{\frac{1}{2}} + y^{\frac{1}{2}})^2$ when $x = 9$ and $y = 4$.
6. Using the values $x = 4$, $y = 3$, show that $(x^2 + y^2)^{-\frac{1}{2}}$ is not equal to $x^{-1} + y^{-1}$.
7. When $x = \frac{4}{9}$, evaluate (a) $9\,x^{\frac{1}{2}}$; (b) $(9\,x)^{-\frac{1}{2}}$; (c) $4\,x^{-\frac{3}{2}}$.
8. Simplify: (a) $(2\,a^{-\frac{3}{4}})^4$; (b) $(4\,x^{-6})^{\frac{1}{2}}$; (c) $(8\,x^{-\frac{3}{4}})^{-\frac{2}{3}}$.
9. Evaluate $(x^{\frac{1}{3}} - y^{\frac{1}{3}})^3$ when $x = 27$ and $y = 8$.
10. Simplify: (a) $(4\,x)^{\frac{1}{2}} \cdot (9\,x)^{-\frac{1}{2}}$; (b) $(4\,x^4)^{\frac{3}{2}} \div (4\,x^{-4})^{-\frac{1}{2}}$.
11. Evaluate: $(\frac{2}{5})^{-2} - (\frac{8}{27})^{-\frac{2}{3}}$.
12. Find the value of x for which $2\,x^{\frac{1}{2}} = \frac{1}{2}$.
13. Find the value of x for which $3\,x^{-3} = -24$.
14. If $2\,x^{-1} = 8$, find the value of $3\,x^{-\frac{1}{2}}$.
15. Solve: (a) $4\,x^{-\frac{2}{3}} = 16$; (b) $(3\,x)^{\frac{3}{4}} = 8$.
16. Simplify: $(-\frac{1}{8}\,a^{-6}b^{12})^{-\frac{1}{3}}$.
17. Simplify: $(3\,x^{-\frac{1}{2}})^2 \cdot (2\,x^{-\frac{2}{3}})^{-3}$.
18. Evaluate $[(2\,x)^{-1} + x^{\frac{1}{2}}]^{-1}$ when $x = \frac{1}{4}$.

Simplify:

19. (a) $\dfrac{x^{-1}}{2\,x^{-2}}$; (b) $\dfrac{4 - x^{-2}}{2 - x^{-1}}$.

20. (a) $\dfrac{x^{\frac{1}{2}} \cdot x^{\frac{1}{6}}}{x^{-\frac{1}{3}}}$; (b) $\dfrac{x^{\frac{3}{2}} - x^{-\frac{3}{2}}}{x^{-\frac{3}{2}}}$.

21. Show that $(x + y)^{-1} \cdot \left(\dfrac{x^{-1} + y^{-1}}{x^{-1}}\right)$ may be reduced to y^{-1}.
22. Express in simplest radical form: $(12\frac{1}{2})^{\frac{1}{2}} - (4\frac{1}{2})^{\frac{1}{2}}$.

23. From the four numbers $16^{-\frac{1}{4}}$, $3^{\frac{1}{2}}$, $8^{-\frac{2}{3}}$, $(-1)^{-3}$ select (**a**) one that is negative; (**b**) one that is irrational; (**c**) one that is the square of one of the others.
24. If $y = x^2$, and $x^3 t = 8$, show that $y = 4\,t^{-\frac{2}{3}}$. Find the value of y in simplest radical form when $t = 2$.
25. If $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 4(x^{\frac{1}{2}} - y^{\frac{1}{2}})$, find the ratio of x to y.
26. Show that $(a^{\frac{1}{2}}b^{-2}c^{\frac{1}{4}})^{-2} \cdot (\frac{1}{2}\,a^{-1}b^2c^{\frac{1}{2}})^{-3}$ may be simplified to
$$\frac{8\,a^2}{b^2c^2}.$$
27. Evaluate $\dfrac{a^{-2} - a^{-1}}{2\,a^{-2}}$ when $a = \frac{3}{4}$.
28. In evaluating $3^{\frac{1}{2}} + 3^{\frac{1}{2}} + 3^{\frac{1}{2}}$, a student treated the expression as though it were $3^{\frac{1}{2}} \cdot 3^{\frac{1}{2}} \cdot 3^{\frac{1}{2}}$, and so obtained the result $3^{\frac{3}{2}}$. (**a**) Is the procedure valid? (**b**) Is the result correct?
29. Evaluate: (**a**) $\dfrac{2^{-1}}{2^{-1}} - \dfrac{2^{-1}}{3^{-1}}$; (**b**) $\dfrac{2^{-1}}{2^{-1} - 3^{-1}}$.
30. Show that $\dfrac{a^2 - a^{-2}}{(a^{\frac{1}{2}} + a^{-\frac{1}{2}})(a^{\frac{1}{2}} - a^{-\frac{1}{2}})}$ may be simplified to the form $a + \dfrac{1}{a}$.

Exercises [B]

1. If $A = \dfrac{a^x + a^{-x}}{2}$, and $B = \dfrac{a^x - a^{-x}}{2}$, show that $A^2 - B^2 = 1$.
2. Simplify, by expressing each factor as a power of 2:
$$2^{-n} \cdot 8^{n-1} \cdot 4^{n+3} \div 16^n.$$
3. (**a**) Divide 10^n by 10. (**b**) Divide 10^n by 100. (**c**) Simplify $\left[\dfrac{10^{n+2}}{100}\right]^{\frac{1}{n}}$.
4. If $x^{\frac{1}{3}}y^{\frac{3}{4}}z^{-\frac{1}{6}} = 3$, express z as a power of 3 when both x and y have value 3.
5. (**a**) Multiply 2^x by 2. (**b**) Show that $2^{x+2} - (2^{x+1} + 2^x) = 2^x$.
6. (**a**) Show that $\dfrac{(x+1)^{\frac{1}{2}} + (x+1)^{-\frac{1}{2}}}{(x+1)^{\frac{1}{2}} - (x+1)^{-\frac{1}{2}}}$ reduces to $\dfrac{x+2}{x}$ for all positive values of x.

   (**b**) Show that $\dfrac{(x+1)^{\frac{1}{2}} + (x-1)^{\frac{1}{2}}}{(x+1)^{\frac{1}{2}} - (x-1)^{\frac{1}{2}}}$ reduces to $x + (x^2-1)^{\frac{1}{2}}$ for all values of x greater than or equal to 1.
7. Show that there are two rational numbers which satisfy the equation $x^{\frac{2}{3}} - 3\,x^{\frac{1}{3}} = 10$.
8. Show that there are four real, but no rational, numbers which satisfy the equation $x^2 + 10\,x^{-2} = 7$.

9. If $x = \frac{1}{2}(y + y^{-1})$, show that $y = x \pm (x^2 - 1)^{\frac{1}{2}}$, and that the two values of y are reciprocal quantities.

10. If $x = \dfrac{3}{t^{\frac{2}{3}} + t^{-\frac{1}{3}}}$, and $y = \dfrac{3}{t^{\frac{1}{3}} + t^{-\frac{2}{3}}}$, show that:
(**a**) $y^3 = tx^3$; (**b**) $x^3 + y^3 = 3\,xy$.

Simplify:

11. $(x-1)^{\frac{1}{2}} + x\,(x-1)^{-\frac{1}{2}}$.

12. $(x^2+2)^{\frac{1}{2}} - x^2\,(x^2+2)^{-\frac{1}{2}}$.

13. $(2x+1)^{\frac{2}{3}} - 4(2x+1)^{-\frac{1}{3}}$.

14. $(1-x^2)^{-\frac{3}{2}} - 3\,x^2\,(1-x^2)^{-\frac{5}{2}}$.

15. $\dfrac{(a^2-x^2)^{\frac{1}{2}} + x^2(a^2-x^2)^{-\frac{1}{2}}}{a^2-x^2}$.

16. $\dfrac{(x+3)^{\frac{1}{4}}}{x} \cdot \left[\dfrac{(x+3)^{\frac{3}{4}} - x(x+3)^{-\frac{1}{4}}}{x+3}\right]$.

	a.	b.	c.	d.
9.	-1	$\frac{1}{4}x^8$	$\frac{3}{x^3}$	$\frac{a^2}{2}$
10.	$\frac{1}{8}x^5$	$2x$	$2x$	2
11.	x^{n+1}	x^{2n}	x^n	x^n
12.	x^{2n}	$x^{(n^2)}$	x^n	1

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	a.	b.	c.	d.
1.	x^4	x^4	$(2x)^4$, or $16x^4$	x^5
2.	x^{10}	x^{16}	x^8	$9x^6$
3.	$15b^8$	$-27b^{15}$	2^{10}, or 1024	x^6
4.	$2n^5$	$16n^{12}$	$-9x^3$	$64x^{12}$
5.	$9x^6$	$6x^3$	1	0
6.	$32y^8$	$12y^4$	2	$4y^4$
7.	$-8p^6$	$-2p^2$	$(-2)^2$, or 4	$2a^4$

8. $x = 18b^2$

9. $a = -\frac{1}{2}c^6$

11. a. 2^{12}. b. 32

12. a. 3^{2n}. b. $n = 6$

Pages 218-220

1. (c). 2. (b). 3. (c). 4. (a). 5. (b). 6. (a). 7. (c). 8. (b).

	a.	b.	c.	d.	e.
9.	9	3	9	$\frac{1}{3}$	1
10.	8	2	8	$\frac{1}{2}$	-2
11.	$\frac{1}{16}$	$\frac{1}{16}$	2	2	2
12.	4	8	$\frac{1}{3}$	1	2
13.	$\frac{1}{8}$	16	9	4	125
14.	27	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{4}{9}$	32
15.	$\frac{1}{4}$	-8	1000	$-\frac{2}{3}$	$\frac{4}{5}$
16.	x^4	x^4	x^3	x^3	
17.	x^3	x^3	x^9	x^9	

	a.	b.	c.	d.
18.	$\frac{1}{x^6}$	$\frac{1}{x^6}$	$\frac{1}{x}$	$\frac{1}{x}$
19.	x	x	x	$\frac{1}{3}x$
20.	x	x^2	x^3	$\frac{1}{4}x$
21.	$\frac{1}{x}$	$6x^{\frac{1}{2}}$	$\frac{1}{2}x^2$	
22.	$\frac{9}{8}$	$\frac{2}{3}$		
23.	1	$x + 1$	$x^2 - 2x + 1$	
24.	x	$x + 1$	$1 - \frac{1}{2}x$	
25.	$\frac{x + 2}{x^2}$	$1 - \frac{1}{2}x$	$\frac{\frac{1}{2}}{\frac{1}{2} + \frac{1}{4}} = \frac{2}{3}$	

26. $x = 4$. 27. $x = 16$. 28. a. $x = \pm\frac{1}{3}$. b. $x = 9$.
29. 3. 30. $\frac{a^{\frac{1}{2}}}{2b^{\frac{1}{2}}}$.

Pages 220-221

1. a. 8
   b. 2
   c. 1
2. a. $\frac{1}{8}$
   b. 10
   c. $\frac{1}{8}$
   d. 2
3. a. 9
   b. $\frac{1}{16}$
   c. 1
4. a. $4p^2q$
   b. $\frac{3a}{b^2}$
   c. $\frac{8b^3}{a^6}$
5. 25
7. a. 6
   b. $\frac{1}{2}$
   c. $\frac{27}{2}$
8. a. $\frac{16}{a^3}$
   b. $\frac{2}{x^3}$
   c. $\frac{x^{\frac{1}{2}}}{4}$
9. 1
10. a. $\frac{2}{3}$
    b. $16x^4$
11. 4
12. $x = \frac{1}{16}$
13. $x = -\frac{1}{2}$
14. 6
15. a. $x = \pm\frac{1}{8}$
    b. $x = 5\frac{1}{3}$
16. $-\frac{2a^2}{b^4}$
17. $\frac{9x}{8}$
18. $\frac{2}{5}$
19. a. $\frac{x}{2}$
    b. $\frac{2x + 1}{x}$
20. a. x
    b. $x^3 - 1$
22. $\sqrt{2}$
23. a. $(-1)^{-3}$
    b. $3^{\frac{1}{2}}$
    c. $8^{-\frac{2}{3}}$
24. $y = 2\sqrt[3]{2}$
25. $\frac{25}{9}$
27. $\frac{1}{8}$
28. a. No. b. Yes
29. a. $-\frac{1}{2}$. b. 3

Pages 221-222

2. 8
3. a. 10^{n-1}
   b. 10^{n-2}
   c. 10
4. $z = 3^{\frac{1}{2}}$
5. a. 2^{x+1}
7. $\{125, -8\}$
8. $\{\sqrt{5}, -\sqrt{5}, \sqrt{2}, -\sqrt{2}\}$
9. $y_1 = x + \sqrt{x^2 - 1}$
   $y_2 = x - \sqrt{x^2 - 1}$

11. $\dfrac{2x - 1}{(x - 1)^{\frac{1}{2}}}$

12. $\dfrac{2}{(x^2 + 2)^{\frac{1}{2}}}$

13. $\dfrac{2x - 3}{(2x + 1)^{\frac{1}{3}}}$

14. $\dfrac{1 - 4x^2}{(1 - x^2)^{\frac{5}{2}}}$

15. $\dfrac{a}{(a^2 - x^2)^{\frac{3}{2}}}$

16. $\dfrac{3}{x(x + 3)}$

Pages 224-225

1. a. 1.1×10^3
   b. 6.2×10^{-1}
   c. 1.86×10^5
   d. 2.5×10^4
   e. 6.1×10^{-2}
   f. 3×10^{10}
   g. 8.5×10^{-5}
   h. 5.97×10^{24}
   i. 5×10^{-17}

2. a. 247,000
   b. 0.000568
   c. 6,800,000,000
   d. 0.502
   e. 0.00468
   f. 15.0
   g. 0.000000452
   h. 711

3. a. $2x^5 + 5x^4 + 3x^2$
   b. $2(10^5) + 5(10^4) + 3(10^2)$
   c. $2503 \times 100 = 250,300$

4. a. $4x^2 + 6$
   b. $4(10^2) + 6$
   c. $406,000 \div 1000 = 406$

5. a. $2x + 1 + 3x^{-1}$
   b. $2(10) + 1 + 3(10^{-1})$
   c. $2.13 \times 10 = 21.3$

6. a. $5x^{-2} + 2x^{-4} + 4x^{-5}$
   b. $5(10^{-2}) + 2(10^{-4}) + 4(10^{-5})$
   c. $5.024 \div 100 = 0.05024$

7. a. 2
   b. -1
   c. 1
   d. -3
   e. 5
   f. -6

8. 630

9. 1600

10. 10,000

11. 450

12. $E = 1.8 \times 10^{14}$

13. a. 1, 10, $\frac{1}{10}$
   b. greater
   c. less
   d. greater

14. a. $0 < x < 1$
   b. $x > 1$
   c. $x < 0$